

GABOR ORTHONORMAL BASES GENERATED BY THE UNIT CUBES

JEAN-PIERRE GABARDO, CHUN-KIT LAI, AND YANG WANG

ABSTRACT. We consider the problem in determining the countable sets Λ in the time-frequency plane such that the Gabor system generated by the time-frequency shifts of the window $\chi_{[0,1]^d}$ associated with Λ forms a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. We show that, if this is the case, the translates by elements Λ of the unit cube in \mathbb{R}^{2d} must tile the time-frequency space \mathbb{R}^{2d} . By studying the possible structure of such tiling sets, we completely classify all such admissible sets Λ of time-frequency shifts when $d = 1, 2$. Moreover, an inductive procedure for constructing such sets Λ in dimension $d \geq 3$ is also given. An interesting and surprising consequence of our results is the existence, for $d \geq 2$, of discrete sets Λ with $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ forming a Gabor orthonormal basis but with the associated “time”-translates of the window $\chi_{[0,1]^d}$ having significant overlaps.

1. INTRODUCTION

Let g be a non-zero function in $L^2(\mathbb{R}^d)$ and let Λ be a discrete countable set on \mathbb{R}^{2d} , where we identify \mathbb{R}^{2d} to the time-frequency plane by writing $(t, \lambda) \in \Lambda$ with $t, \lambda \in \mathbb{R}^d$. The Gabor system associated with the window g consists of the set of translates and modulates of g :

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i \langle \lambda, x \rangle} g(x - t) : (t, \lambda) \in \Lambda\}. \quad (1.1)$$

Such systems were first introduced by Gabor [Gab] who used them for applications in the theory of telecommunication, but there has been a more recent interest in using Gabor system to expand functions both from a theoretical and applied perspective. The branch of Fourier analysis dealing with Gabor systems is usually referred to as Gabor, or time-frequency, analysis. Gröchenig’s monograph [G] provide an excellent and detailed exposition on this subject.

Recall that the Gabor system is a *frame* for $L^2(\mathbb{R}^d)$ if there exists constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{(t, \lambda) \in \Lambda} |\langle f, e^{2\pi i \langle \lambda, \cdot \rangle} g(\cdot - t) \rangle|^2 \leq B\|f\|^2, \quad f \in L^2(\mathbb{R}^d). \quad (1.2)$$

2010 *Mathematics Subject Classification.* Primary 42B05, 42A85.

Key words and phrases. Gabor orthonormal bases, packing, spectral sets, translational tiles, tiling sets.

It is called an orthonormal basis for $L^2(\mathbb{R}^d)$ if it is complete and the elements of the system (1.1) are mutually orthogonal in $L^2(\mathbb{R}^d)$ and have norm 1, or, equivalently, $\|g\| = 1$ and $A = B = 1$ in (1.2). One of the fundamental problems in Gabor analysis is to classify the windows g and time-frequency sets Λ with the property that the associated Gabor system $\mathcal{G}(g, \Lambda)$ forms a (Gabor) frame or an orthonormal basis for $L^2(\mathbb{R}^d)$. This is of course a very difficult problem and only partial results are known. For example, to the best of our knowledge, the complete characterization of time-frequency sets Λ for which (1.1) is a frame for $L^2(\mathbb{R}^d)$ was only done when $g = e^{-\pi x^2}$, the Gaussian window. Lyubarskii, and Seip and Wallsten [L, SW] showed that $\mathcal{G}(e^{-\pi x^2}, \Lambda)$ is a Gabor frame if and only if the lower Beurling density of Λ is strictly greater than 1. If we assume that Λ is a lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, then it is well known that $ab \leq 1$ is a necessary condition for (1.1) to form a frame for $L^2(\mathbb{R}^d)$. Gröchenig and Stöckler [GS] showed that for totally positive functions, (1.1) is a frame if and only if $ab < 1$. If we consider $g = \chi_{[0,c]}$, the characteristic function of an interval, the associated characterization problem is known as the *abc-problem* in Gabor analysis. By rescaling, one may assume that $c = 1$. In that case, the famous Janssen tie showed that the structure of the set of couples (a, b) yielding a frame is very complicated [J1, GH]. A complete solution of the abc-problem was recently obtained by Dai and Sun [DS].

In this paper, we focus our attention on Gabor system of the form (1.1) which yield orthonormal bases for $L^2(\mathbb{R}^d)$. Perhaps the most natural and simplest example of Gabor orthonormal basis is the system $\mathcal{G}(\chi_{[0,1]^d}, \mathbb{Z}^d \times \mathbb{Z}^d)$. The orthonormality property for this system easily follows from that facts that the Euclidean space \mathbb{R}^d can be partitioned by the \mathbb{Z}^d -translates of the hypercube $[0, 1]^d$ and that the exponentials $e^{2\pi i \langle n, x \rangle}$ form an orthonormal basis for the space of square-integrable functions supported on any of these translated hypercubes. A direct generalization of this observation is the following:

Proposition 1.1. *Let $|g| = |K|^{-1/2} \chi_K$, where $|\cdot|$ denotes the Lebesgue measure, and $K \subset \mathbb{R}^d$ is measurable with finite Lebesgue measure. Suppose that*

- *The translates of K by the discrete set \mathcal{J} are pairwise a.e. disjoint and cover \mathbb{R}^d up to a set of zero measure.*
- *For each $t \in \mathcal{J}$, the set of exponentials $\{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda_t\}$ is an orthonormal basis for $L^2(K)$.*

Let

$$\Lambda = \bigcup_{t \in \mathcal{J}} \{t\} \times \Lambda_t. \quad (1.3)$$

Then $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$.

Although its proof is straightforward and will be omitted (see also [LiW]), this proposition gives us a flexible way of constructing large families of Gabor orthonormal basis. The first condition above means that K is a *translational tile* (with \mathcal{J} called an associated *tiling set*) and the second one that $L^2(K)$ admits an orthonormal basis of exponentials. If this last condition holds, K is called a *spectral set* (and each Λ_t is an associated *spectrum*). The connection between translational tiles and spectral sets is quite mysterious. They were in fact conjectured to be the same class of sets by Fuglede [Fu], but that statement was later disproved by Tao [T] and the exact relationship between the two classes remains unclear.

For the fixed window $g_d = \chi_{[0,1]^d}$, we call a countable set $\Lambda \subset \mathbb{R}^{2d}$ *standard* if it is of the form (1.3). Motivated by the complete solution to the *abc*-problem, our main objective in this paper is to characterize the discrete sets Λ (not necessarily lattices) with the property that the Gabor system $\mathcal{G}(g_d, \Lambda)$ is a Gabor orthonormal basis. First, by generalizing the notion of *orthogonal packing region* (see Section 2) in the work of Lagarias, Reeds and Wang [LRW] to the setting of Gabor systems, we deduce a general criterion for $\mathcal{G}(g_d, \Lambda)$ to be a Gabor orthonormal basis.

Theorem 1.2. *$\mathcal{G}(g_d, \Lambda)$ is a Gabor orthonormal basis if and only if $\mathcal{G}(g_d, \Lambda)$ is an orthogonal set and the translates of $[0, 1]^d$ by the elements of Λ tile \mathbb{R}^{2d} .*

This criterion offers a very simple solution to our problem in the one-dimensional case.

Theorem 1.3. *In dimension $d = 1$, the system $\mathcal{G}(g_1, \Lambda)$ is a Gabor orthonormal basis if and only if Λ is standard.*

However, such a simple characterization ceases to exist in higher dimensions. We will introduce an inductive procedure which allows us to construct a Gabor orthonormal basis with window g_d from a Gabor orthonormal basis with window g_n , $n < d$. This procedure can be used to produce many non-standard Gabor orthonormal basis and we call a set Λ obtained through this procedure *pseudo-standard*. Assuming a mild condition on a low-dimensional time-frequency space, we show that $\mathcal{G}(g_d, \Lambda)$ are essentially pseudo-standard (See Theorem 3.6).

Although we do not have a complete description of the sets Λ yielding Gabor orthonormal bases with window g_d in dimension $d \geq 3$, we managed to obtain a complete characterization of those discrete sets $\Lambda \subset \mathbb{R}^4$ such that $\mathcal{G}(g_2, \Lambda)$ form an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem 1.4. *$\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$ if and only if we can partition \mathbb{Z} into \mathcal{J} and \mathcal{J}' such that either*

$$\Lambda = \bigcup_{n \in \mathcal{J}} \{(m + t_{n,k}, n, j + \mu_{k,m,n}, k + \nu_n) : m, j, k \in \mathbb{Z}\} \cup \bigcup_{m \in \mathbb{Z}} \bigcup_{n \in \mathcal{J}'} \{(m + t_n, n)\} \times \Lambda_{m,n}$$

or

$$\Lambda = \bigcup_{m \in \mathcal{J}} \{(m, n + t_{m,j}, j + \nu_m, k + \mu_{j,m,n}) : n, j, k \in \mathbb{Z}\} \cup \bigcup_{n \in \mathbb{Z}} \bigcup_{m \in \mathcal{J}'} \{(m, n + t_m)\} \times \Lambda_{m,n}.$$

where $\Lambda_{m,n} + [0, 1]^2$ tile \mathbb{R}^2 and $t_{n,k}$, $\mu_{k,m,n}$ and ν_n are real numbers in $[0, 1)$ as a function of m, n or k .

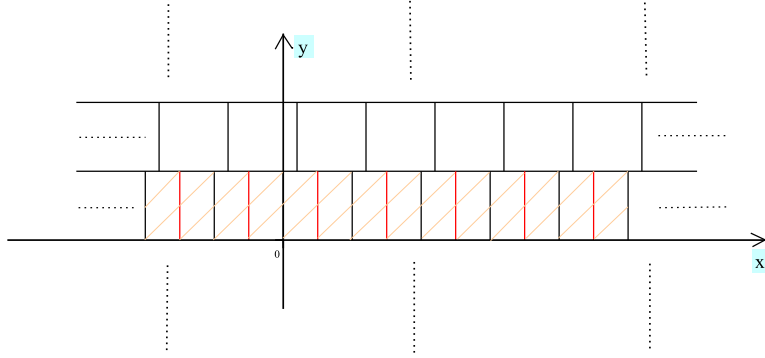


FIGURE 1. This figure illustrates the time-domain of Λ in the first situation of Theorem 1.4. We basically partition \mathbb{R}^2 by horizontal strips. Some strips, like $\mathbb{R} \times [0, 1]$ with $n = 0$, have overlapping structure. This corresponds to the first union of Λ . Some strips, like $\mathbb{R} \times [1, 2]$ with $n = 1$, have tiling structures. This corresponds to the second union of Λ .

We organize the paper as follows. In Section 2, we provide some preliminaries notations and prove Theorem 1.2. In Section 3, we prove Theorem 1.3 and introduce the pseudo-standard time-frequency set. In the last section, we focus on dimension 2 and prove Theorem 1.4.

2. PRELIMINARIES

In this section, we explore the relationship between Gabor orthonormal bases and tilings in the time-frequency space. This theory will be an extension of spectral-tile duality in [LRW] to the setting of Gabor analysis. Denote by $|K|$ the Lebesgue measure of a set K . We say that a closed set T is a *region* if $|\partial T| = 0$ and $\overline{T^\circ} = T$. A bounded region T is called a *translational tile* if we can find a countable set \mathcal{J} such that

- (1) $|(T + t) \cap (T + t')| = 0$, $t, t' \in \mathcal{J}$, $t \neq t'$, and
- (2) $\bigcup_{t \in \mathcal{J}} (T + t) = \mathbb{R}^d$.

In that case, \mathcal{J} is called a *tiling set* for T and $T + \mathcal{J}$ a tiling of \mathbb{R}^d . We will say that $T + \mathcal{J}$ is a packing of \mathbb{R}^n if (1) above is satisfied. We can generalize the notion of tiling and packing to measures and functions. Given a positive Borel measure μ and $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$, the convolution of f and μ is defined to be

$$f * \mu(x) = \int f(x - y) d\mu(y), \quad x \in \mathbb{R}^n,$$

(where a Borel measurable function is chosen in the equivalence class of f to define the integral above). We say that $f + \mu$ is a *tiling* (resp. *packing*) of \mathbb{R}^d if $f * \mu = 1$ (resp. $f * \mu \leq 1$) almost everywhere with respect to the Lebesgue measure. It is clear that if $f = \chi_T$ and $\mu = \delta_{\mathcal{J}}$ where $\delta_{\mathcal{J}} = \sum_{t \in \mathcal{J}} \delta_t$, then $f * \mu = 1$ is equivalent to $T + \mathcal{J}$ being a tiling.

First, we start with the following theorem which gives us a very useful criterion to decide if a packing is actually a tiling. In fact, special cases of this theorem were proved by many different authors in different settings (see e.g. [LRW, Theorem 3.1], [K, Lemma 3.1] and [Li]), but the following version is the most general one as far as we know.

Theorem 2.1. *Suppose that $F, G \in L^1(\mathbb{R}^n)$ are two functions with $F, G \geq 0$ and $\int_{\mathbb{R}^n} F(x) dx = \int_{\mathbb{R}^n} G(x) dx = 1$. Suppose that μ is a positive Borel measure on \mathbb{R}^n such that*

$$F * \mu \leq 1 \quad \text{and} \quad G * \mu \leq 1.$$

*Then, $F * \mu = 1$ if and only if $G * \mu = 1$.*

Proof. By symmetry, it suffices to prove one side of the equivalence. Assuming that $F * \mu = 1$, we have

$$1 = F * \mu \Rightarrow 1 = 1 * G = G * F * \mu = F * G * \mu.$$

Letting $H = G * \mu$ we have $0 \leq H \leq 1$ and $H * F = 1$. We now show that $H = 1$. Indeed letting A be the set $\{x \in \mathbb{R}^n, H(x) < 1\}$ and $B = \mathbb{R}^n \setminus A$, we have

$$(H * F)(x) = \int_{\mathbb{R}^n} H(y) F(x - y) dy = \int_A H(y) F(x - y) dy + \int_B H(y) F(x - y) dy$$

Now, if $|A| > 0$, we have

$$\int_{\mathbb{R}^n} \int_A F(x - y) dy dx = |A| > 0$$

and there exists thus a set E with positive measure such that

$$\int_A F(x - y) dy > 0, \quad x \in E.$$

If $x \in E$, we have

$$\begin{aligned} \int_A H(y) F(x-y) dy + \int_B H(y) F(x-y) dy &< \int_A F(x-y) dy + \int_B F(x-y) dy \\ &= (1 * F)(x) = 1. \end{aligned}$$

This contradicts to the fact that $H * F = 1$ almost everywhere. Hence, $|A| = 0$ and $H = 1$ follows. \square

Let $f, g \in L^2(\mathbb{R}^d)$. We define the *short time Fourier transform* of f with respect to the window g be

$$V_g f(t, \nu) = \int_{\mathbb{R}^{2d}} f(x) \overline{g(x-t)} e^{-2\pi i \langle \nu, x \rangle} dx.$$

Let $\mathcal{G}(g, \Lambda)$ be a Gabor orthonormal basis. Since translating Λ by an element of \mathbb{R}^{2d} does not affect the orthonormality nor the completeness of the given system, there is no loss of generality in assuming that $(0, 0) \in \Lambda$. We say that a region $D (\subset \mathbb{R}^{2d})$ is an *orthogonal packing region* for g if

$$(D^\circ - D^\circ) \cap \mathcal{Z}(V_g g) = \emptyset.$$

Here $\mathcal{Z}(V_g g) = \{(t, \nu) : V_g g(t, \nu) = 0\}$.

Lemma 2.2. *Suppose that $\mathcal{G}(g, \Lambda)$ is a mutually orthogonal set of $L^2(\mathbb{R}^d)$. Let D be any orthogonal packing region for g . Then $\Lambda - \Lambda \subset \mathcal{Z}(V_g g) \cup \{0\}$ and $\Lambda + D$ is a packing of \mathbb{R}^{2d} . Suppose furthermore that $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis. Then $|D| \leq 1$.*

Proof. Let $(t, \lambda), (t', \lambda') \in \Lambda$ be two distinct points in Λ . Then

$$\int g(x-t') \overline{g(x-t)} e^{-2\pi i (\lambda - \lambda') x} dx = 0,$$

or equivalently, after the change of variable $y = x - t'$,

$$\int g(x) \overline{g(x - (t - t'))} e^{-2\pi i (\lambda - \lambda') x} dx = 0.$$

Hence, $V_g g(t - t', \lambda - \lambda') = 0$ and $(t, \lambda) - (t', \lambda') \in \mathcal{Z}(V_g g)$. This means that $(t, \lambda) - (t', \lambda') \notin D^\circ - D^\circ$. Therefore, the intersection of the sets $(t, \lambda) + D$ and $(t', \lambda') + D$ has zero Lebesgue measure.

Suppose now that $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis. Denote by R the diameter of D . By the packing property of $\Lambda + D$,

$$\begin{aligned} |D| \cdot \frac{\#(\Lambda \cap [-T, T]^{2d})}{(2T)^{2d}} &= \frac{1}{(2T)^{2d}} \left| \bigcup_{\lambda \in \Lambda \cap [-T, T]^{2d}} (D + \lambda) \right| \\ &\leq \frac{1}{(2T)^{2d}} |[-T - R, T + R]^{2d}| = \left(1 + \frac{R}{T}\right)^{2d}. \end{aligned}$$

Taking limit $T \rightarrow \infty$ and using the fact that Beurling density of Λ is 1 ([RS]), we have $|D| \leq 1$. \square

We say that an orthogonal packing region D for g is *tight* if we have furthermore $|D| = 1$. We now apply Theorem 2.1 to the Gabor orthonormal basis problem.

Theorem 2.3. *Suppose that $\mathcal{G}(g, \Lambda)$ is an orthonormal set in $L^2(\mathbb{R}^d)$ and that D is a tight orthogonal packing region for g . Then $\mathcal{G}(g, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $\Lambda + D$ is a tiling of \mathbb{R}^{2d} .*

Proof. Let $F = \chi_D$ and $G = |V_g f|^2 / \|f\|_2^2$. Then $\int_{\mathbb{R}^{2d}} F = 1$ and $\int_{\mathbb{R}^{2d}} G = \|g\|_2^2 = 1$. Now, as D is an orthogonal packing region for g , we have in particular

$$\sum_{\lambda \in \Lambda} \chi_D(x - \lambda) \leq 1.$$

This shows that

$$\delta_\Lambda * F = \delta_\Lambda * \chi_D \leq 1.$$

Moreover, $\Lambda + D$ is a tiling of \mathbb{R}^{2d} if and only if $\delta_\Lambda * \chi_D = 1$. On the other hand, (g, Λ) being a mutually orthogonal set, Bessel's inequality yields

$$\sum_{(t, \lambda) \in \Lambda} \left| \int_{\mathbb{R}^d} f(x) \overline{g(x - t)} e^{-2\pi i \langle \lambda, x \rangle} dx \right|^2 \leq \|f\|^2, \quad f \in L^2(\mathbb{R}^d),$$

or, replacing f by $f(x - \tau)e^{2\pi i \nu x}$ with $(\tau, \nu) \in \mathbb{R}^{2d}$,

$$\sum_{(t, \lambda) \in \Lambda} |V_g f(\tau - t, \nu - \lambda)|^2 \leq \|f\|^2, \quad f \in L^2(\mathbb{R}^d).$$

Hence,

$$\delta_\Lambda * G = \delta_\Lambda * \frac{|V_g f|^2}{\|f\|^2} \leq 1$$

with equality if and only if the Gabor orthonormal system is in fact a basis. The conclusion follows then from Theorem 2.1. \square

Proof of Theorem 1.2. Let $g_d = \chi_{[0, 1]^d}$. Using Theorem 2.3, we just need to show that $[0, 1]^{2d}$ is a tight orthogonal packing region for g_d .

We first consider the case $d = 1$. For $g_1 = \chi_{[0,1]}$, a direct computation shows that

$$V_{g_1}g_1(t, \nu) = \begin{cases} 0, & |t| \geq 1; \\ \frac{1}{2\pi i\nu} (e^{2\pi i\nu t} - e^{2\pi i\nu}), & 0 \leq t \leq 1; \\ \frac{1}{2\pi i\nu} (1 - e^{2\pi i\nu(t+1)}), & -1 \leq t \leq 0. \end{cases} \quad (2.1)$$

The zero set of $V_{g_1}g_1$ is therefore given by

$$\mathcal{Z}(V_{g_1}g_1) = \{(t, \nu) : |t| \geq 1\} \cup \{(t, \nu) : \nu(1 - |t|) \in \mathbb{Z} \setminus \{0\}\}. \quad (2.2)$$

Hence, $(0, 1)^2 - (0, 1)^2 = (-1, 1)^2$ does not intersect the zero set and therefore $[0, 1]^2$ is a tight orthogonal packing region for g_1 .

We now consider the case $d \geq 2$. As we can decompose g_d as $\chi_{[0,1]}(x_1)\dots\chi_{[0,1]}(x_d)$, we have

$$V_{g_d}g_d(t, \nu) = V_{g_1}g_1(t_1, \nu_1) \dots V_{g_1}g_1(t_d, \nu_d) \text{ where } t = (t_1, \dots, t_d) \text{ and } \nu = (\nu_1, \dots, \nu_d).$$

The zero set of $V_{g_d}g_d$ is therefore given by

$$\mathcal{Z}(V_{g_d}g_d) = \{(t, \nu) : |t|_{\max} \geq 1\} \cup \left(\bigcup_{i=1}^d \{(t, \nu) : \nu_i(1 - |t_i|) \in \mathbb{Z} \setminus \{0\}\} \right) \quad (2.3)$$

where $|t|_{\max} = \max\{|t_1|, \dots, |t_d|\}$. It follows that $[0, 1]^{2d}$ is a tight orthogonal packing region for g_d . \square

The following example will not be used in later discussion, but it demonstrates the usefulness of the theory for windows other than the unit cube.

Example 2.4. Let $g(x) = \frac{2}{e^{2x} + e^{-2x}}$ be the hyperbolic secant function. It can be shown ([J2]; see also [Ga]) that

$$V_g g(t, \nu) = \frac{\pi \sin(\pi \nu t) e^{-\pi i \nu t}}{\sinh(2t) \sinh(\pi^2 \nu / 2)}$$

and the zero set is given by

$$\mathcal{Z}(V_g g) = \{(t, \nu) : t\nu \in \mathbb{Z} \setminus \{0\}\}.$$

Hence, $[0, 1]^2$ is a tight orthogonal packing region for g . Note that the zero set does not contain any points on the x -axis and y -axis. There is no tiling set Λ for $[0, 1]^2$ such that $\Lambda - \Lambda \subset \mathcal{Z}(V_g g) \cup \{0\}$ (see also Proposition 3.2 in the next section) and thus there is no Gabor orthonormal basis using the hyperbolic secant as a window. This can be viewed as a particular case of a version of the Balian-Low theorem valid for irregular Gabor frames which was recently obtained in [AFK] and which state that Gabor orthonormal bases cannot exist if the window function is in the modulation space $M^1(\mathbb{R}^d)$.

3. GABOR ORTHONORMAL BASES

Using Lemma 2.2, Theorem 1.2 may be restated in the following way:

Theorem 3.1. $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis if and only if the inclusion $\Lambda - \Lambda \subset \mathcal{Z}(V_g g) \cup \{0\}$ holds and $\Lambda + [0, 1]^{2d}$ is a tiling.

In view of the previous result, the possible translational tilings of the unit cube on \mathbb{R}^{2d} play a fundamental role in the solution of our problem. A characterization for these is not available in arbitrary $2d$ dimension but it is easily obtained when $d = 1$. We prove this result here for completeness but it should be well known.

Proposition 3.2. Suppose that $\chi_{[0,1]^2} + \mathcal{J}$ is a tiling of \mathbb{R}^2 with $(0, 0) \in \mathcal{J}$. Then \mathcal{J} is of either of the following two form:

$$\mathcal{J} = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + a_k) \times \{k\} \text{ or } \mathcal{J} = \bigcup_{k \in \mathbb{Z}} \{k\} \times (\mathbb{Z} + a_k) \quad (3.1)$$

where a_k are any real numbers in $[0, 1)$ for $k \neq 0$ and $a_0 = 0$.

Proof. By Keller's criterion for square tilings (see e. g. [LRW, Proposition 4.1]), for any (t_1, t_2) and (t'_1, t'_2) in \mathcal{J} , $t_i - t'_i \in \mathbb{Z} \setminus \{0\}$ for some $i = 1, 2$. Taking $(t'_1, t'_2) = (0, 0)$, we obtain that, for any $(t_1, t_2) \in \mathcal{J} \setminus \{(0, 0)\}$, one of t_1 or t_2 belongs to $\mathbb{Z} \setminus \{0\}$. If $\mathcal{J} \subset \mathbb{Z}$, we must have $\mathcal{J} = \mathbb{Z}$ for $\chi_{[0,1]^2} + \mathcal{J}$ to be tiling of \mathbb{R}^2 and \mathbb{Z} can be written as either of the sets in (3.1) by taking $a_k = 0$ for all k . Suppose that there exists $(s_1, s_2) \in \mathcal{J}$ such that s_1 is not an integer and $s_2 \in \mathbb{Z}$. If $(t_1, t_2) \in \mathcal{J}$ and $t_2 \notin \mathbb{Z}$, then both t_1 and $t_1 - s_1$ must be integers which would imply that s_1 is an integer, contrary to our assumption. Hence, $(s_1, s_2) \in \mathcal{J}$ implies $s_2 \in \mathbb{Z}$ and we can write

$$\mathcal{J} = \bigcup_{k \in \mathbb{Z}} \mathcal{J}_k \times \{k\}.$$

for some discrete set $\mathcal{J}_k \subset \mathbb{R}$. For $\chi_{[0,1]^2} + \mathcal{J}$ to be a tiling of \mathbb{R}^2 , the set \mathcal{J}_k must be of the form $\mathcal{J}_k = \mathbb{Z} + a_k$. In that case \mathcal{J} can be expressed as one of the sets in the first collection appearing in (3.1).

Similarly, if there exists $(s_1, s_2) \in \mathcal{J}$ such that s_2 is not an integer and $s_1 \in \mathbb{Z}$, \mathcal{J} can be expressed as one of the sets in the second collection appearing in (3.1). This completes the proof. \square

We say that the Gabor orthonormal basis $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ is *standard* if

$$\Lambda = \bigcup_{t \in \mathcal{J}} \{t\} \times \Lambda_t,$$

where $\mathcal{J} + [0, 1]^d$ tiles \mathbb{R}^d and Λ_t is a spectrum for $[0, 1]^d$. (Note that, by the result in [LRW], $\Lambda_t + [0, 1]^d$ must then be a tiling of \mathbb{R}^d for every $t \in \mathcal{J}$.)

The following result settles the one-dimensional case.

Theorem 3.3. $\mathcal{G}(\chi_{[0,1]}, \Lambda)$ is a Gabor orthonormal basis if and only if Λ is standard.

Proof. We just need to show that Λ being standard is a necessary condition for $\mathcal{G}(\chi_{[0,1]}, \Lambda)$ to be a Gabor orthonormal basis. We can also assume, for simplicity, that $(0, 0) \in \Lambda$. By Proposition 3.1, if $\mathcal{G}(\chi_{[0,1]}, \Lambda)$ is a Gabor orthonormal basis, then $\Lambda - \Lambda \subset \mathcal{Z}(V_g g) \cup \{0\}$ and $\Lambda + [0, 1]^2$ must be a tiling of \mathbb{R}^2 . By Proposition 3.2, Λ must be of either one of the forms in (3.1). Note that Λ is standard in the second case. In order to deal with the first case, suppose that

$$\Lambda = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + a_k) \times \{k\}, \text{ with } a_k \in [0, 1), \ k \neq 0, \ a_0 = 0.$$

We now show that this is impossible unless $a_k = 0$ for all k (which reduces to the case $\Lambda = \mathbb{Z}^2$, which is standard). We can assume, without loss of generality, that $a_k \neq 0$ for some $k > 0$ with k being the smallest such index. If $a_k \neq 0$ for some k , then both (a_k, k) and $(0, k-1)$ are in Λ . The orthogonality of the Gabor system then implies that $(a_k, 1) \in \mathcal{Z}(V_g g)$. Using (2.2), we deduce that $1 \cdot (1 - |a_k|) \in \mathbb{Z} \setminus \{0\}$. That means a_k must be an integer, which is a contradiction. Hence, the first case is impossible unless $a_k = 0$ for all k and the proof is completed. \square

A description of all time-frequency sets Λ for which $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis however become vastly more complicated when $d \geq 2$. In particular, as we will see, the standard structure cannot cover all possible cases. Consider integers $m, n > 0$ such that $m + n = d$. For convenience and to be consistent with our previous notation, we will write the cartesian product of the two time-frequency spaces \mathbb{R}^{2m} and \mathbb{R}^{2n} in the non-standard form

$$\mathbb{R}^{2d} = \mathbb{R}^{2m} \times \mathbb{R}^{2n} = \{(s, t, \lambda, \nu), (s, \lambda) \in \mathbb{R}^{2m}, (t, \nu) \in \mathbb{R}^{2n}\}.$$

We will also denote by Π_1 the projection operator from \mathbb{R}^{2d} to \mathbb{R}^{2m} defined by

$$\Pi_1((s, t, \lambda, \nu)) = (s, \lambda), \quad (s, t, \lambda, \nu) \in \mathbb{R}^{2d} = \mathbb{R}^{2m} \times \mathbb{R}^{2n}. \quad (3.2)$$

To simplify the notation, we also define $g_k = \chi_{[0,1]^k}$ for any $k \geq 1$. We now build a new family of time-frequency sets on \mathbb{R}^{2d} as follows. Suppose that $\mathcal{G}(\chi_{[0,1]^m}, \Lambda_1)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^m)$ and that we associate with each $(s, \lambda) \in \Lambda_1$, a discrete set $\Lambda_{(s, \lambda)}$ in \mathbb{R}^{2n} such that $\mathcal{G}(\chi_{[0,1]^n}, \Lambda_{(s, \lambda)})$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^n)$. We then define

$$\Lambda = \bigcup_{(s, \lambda) \in \Lambda_1} \{(s, t, \lambda, \nu), (t, \nu) \in \Lambda_{(s, \lambda)}\}. \quad (3.3)$$

We say that a Gabor system $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ with Λ as in (3.3) is *pseudo-standard*.

Proposition 3.4. Every pseudo-standard Gabor system $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^d)$.

Proof. If $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we have $g_d(x, y) = g_m(x)g_n(y)$ (for $m + n = d$). This yields immediately that

$$V_{g_d}g_d(s, t, \lambda, \nu) = V_{g_m}g_m(s, \lambda) V_{g_n}g_n(t, \nu), \quad (s, \lambda) \in \mathbb{R}^{2m}, \quad (t, \nu) \in \mathbb{R}^{2n}. \quad (3.4)$$

Suppose that $\rho = (s, t, \lambda, \nu)$ and $\rho' = (s', t', \lambda', \nu')$ are distinct elements of Λ . If $(s, \lambda) = (s', \lambda')$, then (t, ν) and (t', ν') are distinct elements of $\Lambda_{(s, \lambda)}$ and we have thus

$$(t' - t, \nu' - \nu) \in \mathcal{Z}(V_{g_n}g_n)$$

which implies that $\mathcal{Z}(V_{g_d}g_d)(\rho' - \rho) = 0$. On the other hand, if $(s, \lambda) \neq (s', \lambda')$, we have then

$$(s' - s, \lambda' - \lambda) \in \mathcal{Z}(V_{g_m}g_m)$$

which implies again that $\mathcal{Z}(V_{g_d}g_d)(\rho' - \rho) = 0$. This proves the orthonormality of the system $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$. This proposition can now be proved by invoking Theorem 3.1 if we can show that $\Lambda + [0, 1]^{2d}$ is a tiling of \mathbb{R}^{2d} . To prove this, we note that $\Lambda_1 + [0, 1]^{2m}$ is a tiling of the subspace \mathbb{R}^{2m} by Theorem 3.1 and that, similarly, for each $(t, \lambda) \in \Lambda$, $\Lambda_{(t, \lambda)} + [0, 1]^{2n}$ is a tiling of \mathbb{R}^{2n} . This easily implies the required tiling property and concludes the proof. \square

Example 3.5. Consider the two-dimensional case $d = 2$. Let

$$\Lambda_1 = \bigcup_{m \in \mathbb{Z}} \{m\} \times (\mathbb{Z} + \mu_m), \quad \mu_m \in [0, 1).$$

Associate with each $\gamma = (m, j + \mu_m) \in \Lambda_1$, the set

$$\Lambda_\gamma = \bigcup_{n \in \mathbb{Z}} \{n + s_{m,j}\} \times (\mathbb{Z} + \nu_{n,m,j}), \quad s_{m,j} \in \mathbb{R}, \nu_{n,m,j} \in [0, 1).$$

Then,

$$\Lambda := \{(m, n + s_{m,j}, j + \mu_m, k + \nu_{n,m,j}) : m, n, j, k \in \mathbb{Z}\}$$

(written in the form of $(t_1, t_2, \lambda_1, \lambda_2)$ where (t_1, t_2) are the translations and (λ_1, λ_2) the frequencies) has the pseudo-standard structure. Note that the parameters $s_{m,j}$ can be chosen so that the set Λ is not standard as the set

$$\{(m, n + s_{m,j}), \quad m, n, j \in \mathbb{Z}\} + [0, 1]^2$$

will not tile \mathbb{R}^2 in general. For example, for $m = n = 0$, we could let $s_{0,0} = 0$ and the numbers $s_{0,j}$ could be chosen as distinct numbers in the interval $[0, 1)$. The square $[0, 1]^2$ would then overlap with infinitely many of its translates appearing as part of the Gabor system.

Using a similar procedure to higher dimension, we can produce many non-standard Gabor orthonormal bases with window $\chi_{[0,1]^d}$. However, the pseudo-standard structure still cannot cover all possible cases of time-frequency sets. A time-frequency set could be a mixture of pseudo-standard and standard structure. For example, consider the set

$$\Lambda = \bigcup_{n \in \mathbb{Z} \setminus \{1\}} \{(m + t_{n,k}, n, j + \mu_{k,m,n}, k + \nu_n) : j, k \in \mathbb{Z}\} \cup \{(m, 1)\} \times \Lambda_m,$$

where $\Lambda_m + [0, 1]^2$ tiles \mathbb{R}^2 . This set consists of two parts. The first part is a subset of a set having the pseudo-standard structure while the second part is a subset of a set having the standard one. Moreover, the translates of the unit square associated with the first part are disjoint with those associated with the second part, showing that $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ is a mutually orthogonal set. Since Λ is clearly a tiling of \mathbb{R}^4 , Theorem 3.1 shows that $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ is a Gabor orthonormal basis.

In the next section, we will classify all possible sets $\Lambda \subset \mathbb{R}^4$ with the property that $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$. However, we have

Theorem 3.6. *Let $d = m + n$ and let $\Pi_1 : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2m}$ be defined by (3.2). Suppose that $(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis and that $\Pi_1(\Lambda) + [0, 1]^{2m}$ tiles \mathbb{R}^{2m} . Then Λ has the pseudo-standard structure.*

Proposition 3.7. *Let $d = m + n$ and suppose that $(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. If $(s_0, \lambda_0) \in \mathbb{R}^{2m}$, consider the translate of the unit hypercube in \mathbb{R}^{2m} , $C = (s_0, \lambda_0) + [0, 1]^{2m}$, and define*

$$\Lambda(C) := \{(t, \nu) \in \mathbb{R}^{2n} : (s, t, \lambda, \nu) \in \Lambda \text{ and } (s, \lambda) \in C\}.$$

Then $(\chi_{[0,1]^n}, \Lambda(C))$ is a Gabor orthonormal basis for $L^2(\mathbb{R}^{2n})$.

Proof. We first show that the system $(\chi_{[0,1]^n}, \Lambda(C))$ is orthogonal. Let (t, ν) and (t', ν') be distinct elements of $\Lambda(C)$. There exist (s, λ) and (s', λ') in \mathbb{R}^{2m} such that (s, t, λ, ν) and (s', t', λ', ν') both belong to Λ . Using the mutual orthogonality of the system $(\chi_{[0,1]^d}, \Lambda)$ together with (3.4), we have

$$V_{g_m} g_m(s - s', \lambda - \lambda') = 0 \text{ or } V_{g_n} g_n(t - t', \nu - \nu') = 0.$$

Note that, as both (s, λ) and (s', λ') belong to C , we have $|s - s'|_{\max} < 1$ and $|\lambda - \lambda'|_{\max} < 1$. In particular, $V_{g_m} g_m(s - s', \lambda - \lambda') \neq 0$ and the orthogonality of the system $(\chi_{[0,1]^n}, \Lambda(C))$ follows.

If $(s, \lambda) \in \Pi_1(\Lambda)$ (as defined in (3.2)), let

$$\Lambda_{(s,\lambda)} = \{(t, \nu) : (s, t, \lambda, \nu) \in \Lambda\}.$$

Let $f_1 \in L^2(\mathbb{R}^m)$, $f_2 \in L^2(\mathbb{R}^n)$ and $(s_0, \lambda_0) \in \mathbb{R}^{2m}$. Applying Parseval's identity to the function

$$f(x, y) = e^{2\pi i \lambda_0 \cdot x} f_1(x - s_0) f_2(y), \quad x \in \mathbb{R}^m, \quad y \in \mathbb{R}^n,$$

we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^m} |f_1(x)|^2 dx \int_{\mathbb{R}^n} |f_2(y)|^2 dy \\ &= \sum_{(s, \lambda) \in \Pi_1(\Lambda)} \sum_{(t, \nu) \in \Lambda_{(s, \lambda)}} |V_{g_m} f_1(s - s_0, \lambda - \lambda_0)|^2 |V_{g_n} f_2(t, \nu)|^2 \\ &= \sum_{(s, \lambda) \in \Pi_1(\Lambda)} \sum_{(t, \nu) \in \Lambda_{(s, \lambda)}} |V_{f_1} g_m(s_0 - s, \lambda_0 - \lambda)|^2 |V_{g_n} f_2(t, \nu)|^2 \end{aligned}$$

Defining

$$w(s, \lambda) = \|f_2\|_2^{-2} \sum_{(t, \nu) \in \Lambda_{(s, \lambda)}} |V_{g_n} f_2(t, \nu)|^2 \quad \text{and} \quad \mu = \sum_{(s, \lambda) \in \Pi_1(\Lambda)} w(s, \lambda) \delta_{(s, \lambda)}$$

for $f_2 \neq 0$, the above identity can be written as

$$\int_{\mathbb{R}^m} |f_1(x)|^2 dx = \sum_{(s, \lambda) \in \Pi_1(\Lambda)} w(s, \lambda) |V_{f_1} g_m(s_0 - s, \lambda_0 - \lambda)|^2 = (\mu * |V_{f_1} g_m|^2)(s_0, \lambda_0).$$

On the other hand, letting $\check{\chi}_{[0,1]^{2m}}(s, \lambda) = \chi_{[0,1]^{2m}}(-s, -\lambda)$ and defining C and $\Lambda(C)$ as above, we have also

$$\begin{aligned} (\mu * \check{\chi}_{[0,1]^{2m}})(s_0, \lambda_0) &= \sum_{(s, \lambda) \in \Pi_1(\Lambda)} w(s, \lambda) \chi_{[0,1]^{2m}}(s - s_0, \lambda - \lambda_0) \\ &= \sum_{(s, \lambda) \in \Pi_1(\Lambda) \cap C} w(s, \lambda) \\ &= \|f_2\|_2^{-2} \sum_{(t, \nu) \in \Lambda(C)} |V_{g_n} f_2(t, \nu)|^2 \leq 1, \end{aligned}$$

where the last inequality results from the orthogonality of the system $(\chi_{[0,1]^n}, \Lambda(C))$ proved earlier. Since (s_0, λ_0) is arbitrary in \mathbb{R}^{2m} and

$$\int_{\mathbb{R}^{2m}} |V_{f_1} g_m(s, \lambda)|^2 ds d\lambda = \|f_1\|_2^2,$$

Theorem 2.1 can be used to deduce that $\mu * \check{\chi}_{[0,1]^m} = 1$. This shows that

$$\sum_{(t, \nu) \in \Lambda(C)} |V_{g_n} f_2(t, \nu)|^2 = \|f_2\|^2, \quad f_2 \in L^2(\mathbb{R}^n),$$

and thus that the system $(\chi_{[0,1]^n}, \Lambda(C))$ is complete, proving our claim. \square

Proof of Theorem 3.6. Let $\mathcal{J} = \Pi_1(\Lambda)$ and, for any $(s, \lambda) \in \mathcal{J}$, define

$$\Lambda_{(s, \lambda)} = \{(t, \nu) : (s, t, \lambda, \nu) \in \Lambda\}.$$

If $(s_0, \lambda_0) \in \mathcal{J}$, let $C = (s_0, \lambda_0) + [0, 1)^{2m}$, and

$$\Lambda(C) := \{(t, \nu) \in \mathbb{R}^{2n} : (s, t, \lambda, \nu) \in \Lambda \text{ and } (s, \lambda) \in C\}.$$

Proposition 3.7 shows that the system $(\chi_{[0,1]^n}, \Lambda(C))$ forms a Gabor orthonormal basis. By assumption $\mathcal{J} + [0, 1)^{2m}$ tiles \mathbb{R}^{2m} . Hence, $(s_0, \lambda_0) + [0, 1)^{2m}$ contains exactly one point in \mathcal{J} , i.e. (s_0, λ_0) , and we have

$$\Lambda(C) = \{(t, \nu) : (s_0, t, \lambda_0, \nu) \in \Lambda\} = \Lambda_{(s_0, \lambda_0)}.$$

Therefore, we can write Λ as

$$\Lambda = \bigcup_{(s_0, \lambda_0) \in \mathcal{J}} \{(s_0, \lambda_0)\} \times \Lambda_{(s_0, \lambda_0)}.$$

Our proof will be complete if we can show that \mathcal{J} is a Gabor orthonormal basis of $L^2(\mathbb{R}^m)$.

As \mathcal{J} is a tiling set, by Proposition 3.1 it suffices to show that the inclusion $\mathcal{J} - \mathcal{J} \subset \mathcal{Z}(V_{g_m}g_m) \cup \{0\}$ holds. Let (s, λ) and (s', λ') be distinct points in \mathcal{J} . As $\Lambda_{(s, \lambda)} + [0, 1)^{2n}$ tiles \mathbb{R}^{2n} , so does $\Lambda_{(s, \lambda)} + [-1, 0)^{2n}$, and we can find $(t, \nu) \in \Lambda_{(s, \lambda)}$ such that $0 \in (t, \nu) + [-1, 0)^{2n}$, or, equivalently, with $(t, \nu) \in [0, 1)^{2n}$. Similarly, we can find $(t', \nu') \in \Lambda_{(s', \lambda')}$ such that $(t', \nu') \in [0, 1)^{2n}$. Using the fact that $(\chi_{[0,1]^d}, \Lambda)$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^{2d})$, we have

$$(s, t, \lambda, \nu) - (s', t', \lambda', \nu') \in \mathcal{Z}(V_{g_d}g_d).$$

or, equivalently,

$$V_{g_m}g_m(s - s', \lambda - \lambda') = 0 \quad \text{or} \quad V_{g_n}g_n(t - t', \nu - \nu') = 0.$$

Note that, since $|t - t'| < 1$ and $|\nu - \nu'| < 1$, $V_{g_n}g_n(t - t', \nu - \nu') \neq 0$. Hence $(s, \lambda) - (s', \lambda') \in \mathcal{Z}(V_{g_m}g_m)$ as claimed. \square

4. TWO-DIMENSIONAL GABOR ORTHONORMAL BASES

In this section, our goal will be to classify all possible Gabor orthonormal basis generated by the unit square on \mathbb{R}^2 .

Given a fixed Gabor orthonormal basis $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ and a set $A \subset \mathbb{R}^2$, we define the sets

$$\Gamma(A) = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : (t_1, t_2, \lambda_1, \lambda_2) \in \Lambda, (t_1, t_2) \in A\},$$

and, for any $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and any set $B \subset \mathbb{R}^2$, we let

$$T_A(\lambda_1, \lambda_2) = \{(t_1, t_2) \in \mathbb{R}^2 : (t_1, t_2, \lambda_1, \lambda_2) \in \Lambda, (t_1, t_2) \in A\}$$

and

$$T_A(B) = \{(t_1, t_2) \in \mathbb{R}^2 : (t_1, t_2, \lambda_1, \lambda_2) \in \Lambda, (t_1, t_2) \in A, (\lambda_1, \lambda_2) \in B\}.$$

In particular, the set $T_A(\Gamma(A))$ collects all the couples $(t_1, t_2) \in A$ such that $(t_1, t_2, \lambda_1, \lambda_2) \in \Lambda$ for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

We say that a square is *half-open* if it is a translate of one of the sets

$$[0, 1]^2, \quad (0, 1]^2, \quad [0, 1) \times (0, 1] \quad \text{or} \quad (0, 1] \times [0, 1).$$

Two measurable subsets of \mathbb{R}^d will be called *essentially disjoint* if their intersection has zero Lebesgue measure. In the derivation below, we will make use of the identity

$$V_{g_2}g_2(t_1, t_2, \lambda_1, \lambda_2) = V_{g_1}g_1(t_1, \lambda_1)V_{g_1}g_1(t_2, \lambda_2), \quad (t_1, t_2, \lambda_1, \lambda_2) \in \mathbb{R}^4,$$

which implies, in particular, that

$$V_{g_2}g_2(t_1, t_2, \lambda_1, \lambda_2) = 0 \iff V_{g_1}g_1(t_1, \lambda_1) = 0 \text{ or } V_{g_1}g_1(t_2, \lambda_2) = 0.$$

Moreover, using (2.3), the zero set of $V_{g_2}g_2$ is given by

$$\mathcal{Z}(V_{g_2}g_2) = \{(t, \lambda) : |t|_{\max} \geq 1\} \cup \left(\bigcup_{i=1}^2 \{(t, \nu) : \lambda_i(1 - |t_i|) \in \mathbb{Z} \setminus \{0\}\} \right). \quad (4.1)$$

This implies that if $|t|_{\max} < 1$ and $(t, \lambda) \in \mathcal{Z}(V_{g_2}g_2)$, then, there exists $i \in \{1, 2\}$ and for some integer $m \neq 0$ such that

$$|\lambda_i| = \frac{|m|}{1 - |t_i|} \geq 1.$$

with a strict inequality if $t_i \neq 0$. These properties will be used throughout this section.

Lemma 4.1. *Let $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ be a Gabor orthonormal basis for $L^2(\mathbb{R}^2)$ and let C be a half-open square. Then,*

- (i) $\Gamma(C) + [0, 1]^2$ is a packing of \mathbb{R}^2 .
- (ii) If $(\lambda_1, \lambda_2) \in \Gamma(C)$, then $T_C(\lambda_1, \lambda_2)$ consists of one point.

Proof. (i) Let (λ_1, λ_2) and (λ'_1, λ'_2) be distinct elements of $\Gamma(C)$. By definition, we can find (t_1, t_2) and (t'_1, t'_2) in C such that $(t_1, t_2, \lambda_1, \lambda_2), (t'_1, t'_2, \lambda'_1, \lambda'_2) \in \Lambda$. We then have

$$0 = V_{g_1}g_1(t_1 - t'_1, \lambda_1 - \lambda'_1) V_{g_1}g_1(t_2 - t'_2, \lambda_2 - \lambda'_2)$$

If, without loss of generality, the first factor on the right-hand side of the previous equality vanishes, the fact that $|t_1 - t'_1| < 1$ shows the existence of an integer $k > 0$ such that

$$|\lambda_1 - \lambda'_1| = k/(1 - |t_1 - t'_1|) \geq 1.$$

Hence, the cubes $(\lambda_1, \lambda_2) + [0, 1]^2$ and $(\lambda'_1, \lambda'_2) + [0, 1]^2$ are essentially disjoint.

- (ii) Suppose that $T_C(\lambda_1, \lambda_2)$ contains two distinct points (t_1, t_2) and (t'_1, t'_2) . Then,

$$0 = V_{g_1}g_1(t_1 - t'_1, 0) V_{g_1}g_1(t_2 - t'_2, 0).$$

As $V_{g_1}g_1(t, 0) \neq 0$ for any t with $|t| < 1$, we must have $|t_1 - t'_1| \geq 1$ or $|t_2 - t'_2| \geq 1$, contradicting the fact that both (t_1, t_2) and (t'_1, t'_2) belong to C . \square

In the following, we will denote by ∂A the boundary of a set A . The next result will be useful.

Lemma 4.2. *Under the hypotheses of the previous lemma, consider an element $\lambda = (\lambda_1, \lambda_2)$ of $\Gamma(C)$ and let $T_C(\lambda) = \{(t_1, t_2)\}$. Then for any $x \in \partial(\lambda + [0, 1]^2)$, we can find $\lambda_x = (\lambda_{1,x}, \lambda_{2,x}) \in \Gamma(C)$ such that $x \in \partial(\lambda_x + [0, 1]^2)$. Moreover, for any such λ_x , letting $T_C(\lambda_x) = \{t_x\}$, where $t_x = (t_{1,x}, t_{2,x})$, we can find $i_0 \in \{1, 2\}$ such that $t_{i_0,x} = t_{i_0}$ and $\lambda_{i_0,x} = \lambda_{i_0} + 1$ or $\lambda_{i_0} - 1$.*

Proof. We can write $x = (\lambda_1 + \epsilon_1, \lambda_2 + \epsilon_2)$, where $0 \leq \epsilon_i \leq 1$, $i = 1, 2$ and $\epsilon_i \in \{0, 1\}$ for at least one index i . Let $a = (a_1, a_2) \in \mathbb{R}^2$ with $0 < a_i < 1$ for $i = 1, 2$ and consider the point $(t_a, x) := (t_1 + a_1, t_2 + a_2, \lambda_1 + \epsilon_1, \lambda_2 + \epsilon_2)$ in \mathbb{R}^4 . Since $\Lambda + [0, 1]^4$ is a tiling on \mathbb{R}^4 and the point (t_a, x) is a point on the boundary of $(t, \lambda) + [0, 1]^4$, we can find some point $(t_{x,a}, \lambda_{x,a}) \in \Lambda \setminus \{(t, \lambda)\}$ such that $(t_a, x) \in (t_{x,a}, \lambda_{x,a}) + [0, 1]^4$. Let $t_{x,a} = (t'_1, t'_2)$ and $\lambda_{x,a} = (\lambda'_1, \lambda'_2)$. We have

$$\begin{cases} -a_i \leq t_i - t'_i \leq 1 - a_i, \\ -\epsilon_i \leq \lambda_i - \lambda'_i \leq 1 - \epsilon_i, \end{cases} \quad i = 1, 2 \quad (4.2)$$

Using the orthogonality of the system $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$, we can find $i_0 \in \{1, 2\}$ such that $V_{g_1}g_1(t_{i_0} - t'_{i_0}, \lambda_{i_0} - \lambda'_{i_0}) = 0$. Note that $t_{i_0} - t'_{i_0} \neq 0$ would imply that $|\lambda_{i_0} - \lambda'_{i_0}| > 1$ which is impossible from (4.2). Hence, $t_{i_0} = t'_{i_0}$ and $\lambda_{i_0} - \lambda'_{i_0} \neq 0$.

Moreover, as $V_{g_1}g_1(0, v) \neq 0$ if $|v| < 1$, $V_{g_1}g_1(t_{i_0} - t'_{i_0}, \lambda_{i_0} - \lambda'_{i_0}) = 0$ can only occurs if $|\lambda_{i_0} - \lambda'_{i_0}| = 1$. This shows also that $\epsilon_{i_0} \in \{0, 1\}$ in that case. This proves the last statement of our claim and the fact that $x \in \partial(\lambda_{x,a} + [0, 1]^2)$. The proof will be complete if we can show that $\lambda_{x,a} \in \Gamma(C)$ for some choice of a .

For simplicity, we consider the half-open square to be $C = [b_1, b_1 + 1) \times [b_2, b_2 + 1)$. Our assertion will be true if the point $t_{x,a} = (t'_1, t'_2)$ constructed above satisfies the inequalities $b_i \leq t'_i < b_i + 1$ for $i = 1, 2$. As $t_{i_0} = t'_{i_0}$, the inequalities clearly hold for $i = i_0$. Suppose that the other index j falls out of the range, say $t'_j < b_j$ (The case $t'_j \geq b_j + 1$ is similar). We consider $(t_{a'}, x)$ with $a'_j = t'_j + 1 - t_j + \delta$ for some small $\delta > 0$. Note that, by (4.2), we have $t_i + a_i - 1 \leq t'_i \leq t_i + a_i$ for $i = 1, 2$, and, in particular,

$$a'_j = t'_j + 1 - t_j + \delta \geq a_j + \delta > 0.$$

We have also $a'_j < 1$. Indeed, the inequality $t'_j - t_j + 1 + \delta \geq 1$ would imply that $t'_j + 1 + \delta \geq 1 + t_j$. This is not possible, as $b_j \leq t_j < b_j + 1$, so $1 + t_j \geq b_j + 1$. But $t'_j < b_j$, so $t'_j + 1 < b_j + 1$, so for δ small,

$$t'_j + 1 + \delta < b_j + 1 \leq 1 + t_j$$

which yields a contradiction.

Using the previous argument with a' replacing a , we guarantee the existence of t''_j such that $t'_j + \delta = t_j + a'_j - 1 \leq t''_j \leq t_j + a'_j = t'_j + 1 + \delta$ and the associated point

$(t_{a'}, \lambda_{x,a'}) = (t''_1, t''_2, \lambda''_1, \lambda''_2)$ in Λ with the property that $x \in \partial(\lambda_{x,a'} + [0, 1]^2)$ for some index i'_0 such that $|\lambda_{i'_0} - \lambda''_{i'_0}| = 1$, $t_{i'_0} = t''_{i'_0}$ and $\epsilon_{i'_0} \in \{0, 1\}$. We claim that $t''_j = t'_j + 1$. Now, $(t'_1, t'_2, \lambda'_1, \lambda'_2)$ and $(t''_1, t''_2, \lambda''_1, \lambda''_2)$ are in Λ . The mutual orthogonality property implies that $V_{g_1}g_1(t'_i - t''_i, \lambda'_i - \lambda''_i) = 0$ for some $i = 1, 2$.

Suppose that x is not of the corner points of $\lambda + [0, 1]^2$. In that case, the index i such that $\epsilon_i \in \{0, 1\}$ is unique and it follows that $i_0 = i'_0$. This implies in particular, that $t'_{i_0} = t''_{i_0}$ (as $t'_{i_0} = t_{i_0} = t_{i'_0} = t''_{i'_0} = t''_{i_0}$). Furthermore, the second set of inequalities in (4.2) show that $\lambda'_{i_0} = \lambda''_{i_0} = \lambda_{i_0} - 1$ if $\epsilon_{i_0} = 0$ and $\lambda'_{i_0} = \lambda''_{i_0} = \lambda_{i_0} + 1$ if $\epsilon_{i_0} = 1$. We have thus $\lambda'_{i_0} = \lambda''_{i_0}$ in both cases. We have thus

$$V_{g_1}g_1(t'_{i_0} - t''_{i_0}, \lambda'_{i_0} - \lambda''_{i_0}) = V_{g_1}g_1(0, 0) = 1.$$

Therefore, the other index j must satisfy $V_{g_1}g_1(t'_j - t''_j, \lambda'_j - \lambda''_j) = 0$. The inequalities

$$-\epsilon_j \leq \lambda_j - \lambda'_j \leq 1 - \epsilon_j \quad \text{and} \quad -\epsilon_j \leq \lambda_j - \lambda''_j \leq 1 - \epsilon_j$$

yield $-1 \leq \lambda'_j - \lambda''_j \leq 1$. However, $\delta \leq t''_j - t'_j \leq 1 + \delta$. The $V_{g_1}g_1$ would not be zero unless $t''_j \geq t'_j + 1 (\geq b_j)$. Hence, $t'_j + 1 \leq t''_j \leq t'_j + 1 + \delta$. This forces that $t''_j = t'_j + 1$. This completes the proof for non-corner points. If x is of the corner point, as the square constructed for the non-corner will certainly cover the corner point. Therefore, the proof is completed. \square

With the help of the previous two lemmas, the following tiling result for $\Gamma(C)$ follows immediately.

Corollary 4.3. *Let C be a half-open square. Then $\Gamma(C) + [0, 1]^2$ is a tiling of \mathbb{R}^2 .*

Proof. It suffices to prove the following statement: suppose that $\mathcal{J} + [0, 1]^2$ is non-empty packing of \mathbb{R}^2 . If, for any $x \in \partial(t + [0, 1]^2)$ where $t \in \mathcal{J}$, we can find $t_x \in \mathcal{J}$ with $t_x \neq t$ such that $x \in \partial(t_x + [0, 1]^2)$, then $\mathcal{J} + [0, 1]^2$ is a tiling of \mathbb{R}^2 . Indeed, by Lemma 4.1(i) and Lemma 4.2, $\Gamma(C) + [0, 1]^2$ is a packing of \mathbb{R}^2 and satisfies the stated property. It is thus a tiling of \mathbb{R}^2 .

To prove the previous statement, we note that as $\mathcal{J} + [0, 1]^2$ is packing, it is a closed set. Suppose that $\mathcal{J} + [0, 1]^2$ satisfies the property above and that $\mathbb{R}^d \setminus (\mathcal{J} + [0, 1]^2) \neq \emptyset$. Let $x \in \partial(\mathcal{J} + [0, 1]^2)$ and assume that $x \in t + [0, 1]^2$. We can then find $t_x \in \mathcal{J}$ with $t_x \neq t$ such that $x \in \partial(t_x + [0, 1]^2)$. Note that if x were not a corner point of either $t + [0, 1]^2$ or $t_x + [0, 1]^2$, then x would be in the interior of $\mathcal{J} + [0, 1]^2$. Hence, x must be a corner point of $t + [0, 1]^2$ or $t_x + [0, 1]^2$. As the set of all the corner points of the squares in $\mathcal{J} + [0, 1]^2$ is countable, the Lebesgue measure of the open set $\mathbb{R}^d \setminus (\mathcal{J} + [0, 1]^2)$ is zero and $\mathbb{R}^d \setminus (\mathcal{J} + [0, 1]^2)$ is thus empty, proving our claim. \square

Lemma 4.4. *Let C be a half-open square and suppose that $(\lambda_1, \lambda_2) \in \Gamma(C)$ with $T_C(\lambda_1, \lambda_2) = \{(t_1, t_2)\}$. Then all the sets $T_C(\lambda'_1, \lambda'_2)$ with $(\lambda'_1, \lambda'_2) \in \Gamma(C)$ are either of the form $\{(t_1, t_2 + s)\}$ or $\{(t_1 + s, t_2)\}$ for some real s with $|s| < 1$ depending on (λ_1, λ_2) .*

Proof. We first make the following remark. If $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma(C)$ are such that the two squares $(\alpha_1, \alpha_2) + [0, 1]^2$ and $(\beta_1, \beta_2) + [0, 1]^2$ intersect each other and also both intersect a third square $(\gamma_1, \gamma_2) + [0, 1]^2$ with $(\gamma_1, \gamma_2) \in \Gamma(C)$, then, letting $T_C(\gamma_1, \gamma_2) = \{(r_1, r_2)\}$, we have

$$T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\} \quad \text{and} \quad T_C(\beta_1, \beta_2) = \{(r_1 + b, r_2)\}$$

or

$$T_C(\alpha_1, \alpha_2) = \{(r_1, r_2 + a)\} \quad \text{and} \quad T_C(\beta_1, \beta_2) = \{(r_1, r_2 + b)\},$$

for some real a, b . Indeed, using Lemma 4.2, we have $T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\}$ or $\{(r_1, r_2 + a)\}$ and $T_C(\beta_1, \beta_2) = \{(r_1 + b, r_2)\}$ or $\{(r_1, r_2 + b)\}$. Suppose, for example, that $T_C(\alpha_1, \alpha_2) = \{(r_1 + a, r_2)\}$ and $T_C(\beta_1, \beta_2) = \{(r_1, r_2 + b)\}$. Since the two squares intersect each other, we must have $|\alpha_1 - \beta_1| \leq 1$ and $|\alpha_2 - \beta_2| \leq 1$. The orthogonality property also implies that either $(a, \alpha_1 - \beta_1)$ or $(-b, \alpha_2 - \beta_2)$ is in the zero set of $V_{g_1} g_1$. But since we have $|a|, |b| < 1$, this would imply that $|\alpha_1 - \beta_1| > 1$ or $|\alpha_2 - \beta_2| > 1$, which cannot happen. As $\Gamma(C) + [0, 1]^2$ is a tiling of \mathbb{R}^2 , for any square $(\sigma_1, \sigma_2) + [0, 1]^2$ intersecting the square $(\lambda_1, \lambda_2) + [0, 1]^2$ and with $(\sigma_1, \sigma_2) \in \Gamma(C)$, we can find another square $(\delta_1, \delta_2) + [0, 1]^2$, with $(\delta_1, \delta_2) \in \Gamma(C)$ and with $(\delta_1, \delta_2) + [0, 1]^2$ intersecting both squares $(\sigma_1, \sigma_2) + [0, 1]^2$ and $(\lambda_1, \lambda_2) + [0, 1]^2$. By the previous remark, the conclusion of the lemma holds for all the squares that neighbour the square $(\lambda_1, \lambda_2) + [0, 1]^2$. Replacing this original square by one of the neighbouring squares and continuing this process, we obtain the conclusion of the lemma for all the squares in the tiling $\Gamma(C) + [0, 1]^2$ by an induction argument. This proves our claim. \square

Suppose that the system $\mathcal{G}(\chi_{[0,1]^2}, \Lambda)$ gives rise to a non-standard Gabor orthonormal basis of $L^2(\mathbb{R}^2)$. Then, some of the squares will have overlaps and, without loss of generality, we can assume that

$$|[0, 1]^2 \cap [0, 1]^2 + (t_1, t_2)| > 0$$

for some (t_1, t_2) in the translation component of Λ .

Lemma 4.5. *If $(0, 0, 0, 0) \in \Lambda$, then the sets $T_{[0,1]^2}(\lambda_1, \lambda_2)$ where $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$ are either all of the form $\{(t, 0)\}$ or all of the form $\{(0, t)\}$ with some t (depending on (λ_1, λ_2)) with $|t| < 1$. In the first case, if there exists some $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$ with $T_{[0,1]^2}(\lambda_1, \lambda_2) = (t, 0)$ and $t \neq 0$, then*

$$\Gamma([0, 1]^2) = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + \mu_{k,0}) \times \{k\} \quad (4.3)$$

for some $0 \leq \mu_{k,0} < 1$. Moreover, we can find $0 \leq t_k < 1$ such that

$$T_{[0,1]^2}((\mathbb{Z} + \mu_{k,0}) \times \{k\}) = \{(t_k, 0)\}, \quad k \in \mathbb{Z}, \quad (4.4)$$

and

$$\Lambda \cap ([0, 1]^2 \times \mathbb{R}^2) = \{(t_k, 0, j + \mu_{k,0}, k) : j, k \in \mathbb{Z}\}. \quad (4.5)$$

(In the second case, $\Gamma([0, 1]^2) = \bigcup_{k \in \mathbb{Z}} \{k\} \times (\mathbb{Z} + \mu_{k,0})$ and $T_{[0,1]^2}(\{k\} \times (\mathbb{Z} + \mu_{k,0})) = \{(0, t_k)\}$, $\Lambda \cap ([0, 1]^2 \times \mathbb{R}^2) = \{(0, t_k, k, j + \mu_{k,0}) : j, k \in \mathbb{Z}\}$).

Proof. If $\lambda = (0, 0)$, we have $T_{[0,1]^2}(\lambda) = \{(0, 0)\}$ as $(0, 0, 0, 0) \in \Lambda$. By Lemma 4.4, any $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$ with the square $(\lambda_1, \lambda_2) + [0, 1]^2$ intersecting $[0, 1]^2$ on the λ_1, λ_2 -plane satisfies $T_{[0,1]^2}(\lambda_1, \lambda_2) = \{(t, 0)\}$ or $T_{[0,1]^2}(\lambda_1, \lambda_2) = \{(0, t)\}$ with $|t| < 1$. Without loss of generality, we assume that the first case holds. As $\Gamma([0, 1]^2) + [0, 1]^2$ is a tiling of \mathbb{R}^2 , for any square $C = (\lambda_1, \lambda_2) + [0, 1]^2$, with $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$, we can find squares $C_i = (\lambda_{1,i}, \lambda_{2,i}) + [0, 1]^2$ for $i = 0, \dots, k$ with $(\lambda_{1,i}, \lambda_{2,i}) \in \Gamma([0, 1]^2)$ and such that $C_0 = [0, 1]^2$, $C_k = C$, and with C_i and C_{i+1} touching each other for all $i = 0, \dots, k-1$.

We have $T_{[0,1]^2}(\lambda_{1,1}, \lambda_{2,1}) = \{(t_1, 0)\}$ for some number t_1 with $|t_1| < 1$. Since C_2 and C_0 both intersect C_1 , $T_{[0,1]^2}(\lambda_{1,2}, \lambda_{2,2}) = \{(t_2, 0)\}$ by Lemma 4.4 again. Inductively, we have $T_{[0,1]^2}(\lambda_{1,i}, \lambda_{2,i}) = \{(t_i, 0)\}$, $i = 1, \dots, k$, which proves the first part.

Consider the case where, for any $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$, there exists a number $t = t(\lambda_1, \lambda_2)$ such that $T_{[0,1]^2}(\lambda_1, \lambda_2) = \{(t, 0)\}$ and assume that $t(\lambda_1, \lambda_2) \neq 0$ for at least one couple $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$. Suppose that $\Gamma([0, 1]^2)$ is not of the form in (4.3). By Corollary 4.3 and Proposition 3.2, we must have $\Gamma([0, 1]^2) = \bigcup_{k \in \mathbb{Z}} \{k\} \times (\mathbb{Z} + a_k)$ with $0 \leq a_k < 1$ and at least one $a_k \neq 0$. Consider the distinct points

$$(t, 0, k, a_k + j) \quad \text{and} \quad (t', 0, k-1, a_{k-1} + j), \quad \text{both in } \Lambda.$$

We must have that either $(t-t', 1) \in \mathcal{Z}(V_{g_1}g_1)$ or $(0, a_k - a_{k-1}) \in \mathcal{Z}(V_{g_1}g_1)$. However, since $|a_k - a_{k-1}| < 1$, the second case is impossible. This means that $(t-t', 1) \in \mathcal{Z}(V_{g_1}g_1)$ which is possible only if $t = t'$. Therefore the fact that $(t, 0, k, a_k + j) \in \Lambda$ implies that $t = t_j$ for some real t_j . We know prove by induction on $|j|$ that $t_j = 0$ for all $j \in \mathbb{Z}$. The case $j = 0$ is clear as $(0, 0, 0, 0) \in \Lambda$ by assumption. If our claim is true for all $|j| \leq J$ where $J \geq 0$, chose $k \in \mathbb{Z}$ such that $a_{k+1} \neq 0$ and $a_k = 0$ if such k exists. Suppose first that $j > 0$. There exist thus $t \in [0, 1)$ such that

$$(t_{j+1}, 0, k, j+1) \quad \text{and} \quad (0, 0, k+1, a_{k+1} + j) \quad \text{both belong to } \Lambda.$$

This implies that either $(t, -1) \in \mathcal{Z}(V_{g_1}g_1)$ or $(0, a_{k+1} - 1) \in \mathcal{Z}(V_{g_1}g_1)$. This last case is impossible and the first one is only possible if $t = 0$, showing that $t_{j+1} = 0$. Similarly by considering the points

$$(t_{j-1}, 0, k+1, a_{k+1} + j-1) \quad \text{and} \quad (0, 0, k, j) \quad \text{which both belong to } \Lambda.$$

we can conclude that $t_{j-1} = 0$ for $j < 0$. If k as above does not exist, there exists chose $k' \in \mathbb{Z}$ such that $a_{k'-1} \neq 0$ and $a_{k'} = 0$. By considering the points

$$(t_{j+1}, 0, k', j+1) \quad \text{and} \quad (0, 0, k' - 1, a_{k'-1} + j) \quad \text{if } j > 0$$

and the points

$$(t_{j-1}, 0, k' - 1, a_{k'-1} + j - 1) \quad \text{and} \quad (0, 0, k', j) \quad \text{if } j < 0$$

which all belong to Λ , we conclude that $t_j = 0$ if $|j| = J + 1$. This proves (4.3).

If we are in the first case, i.e.

$$\Gamma([0, 1]^2) = \bigcup_{k \in \mathbb{Z}} (\mathbb{Z} + \mu_{k,0}) \times \{k\},$$

let m, m' be distinct integers. We have then

$$T_{[0,1]^2}(m + \mu_{n,0}, n) = \{(t_m, 0)\} \quad \text{and} \quad T_{[0,1]^2}(m' + \mu_{n,0}, n) = \{(t_{m'}, 0)\}$$

which implies that $V_{g_1}g_1(t_m - t_{m'}, m - m') = 0$ or $V_{g_1}g_1(0, 0) = 0$. The second case is clearly impossible while the first one is possible only when $t_m = t_{m'}$. This shows (4.4) and (4.5) follows immediately from (4.3) and (4.4). \square

Note that Lemma 4.5 implies that $\Gamma([0, 1]^2) = \Gamma(\{(x, 0) : 0 \leq x < 1\})$ and $\Gamma((0, 1)^2) = \emptyset$ if $(0, 0, 0, 0) \in \Lambda$.

Lemma 4.6. *Under the assumptions of Lemma 4.5, suppose that there exists $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$ with $T_{[0,1]^2}(\lambda_1, \lambda_2) = (t, 0)$ and $t \neq 0$. Then we can find numbers t_k with $0 \leq t_k < 1$ and $\mu_{k,m}$, $k, m \in \mathbb{Z}$, with $0 \leq \mu_{k,m} < 1$, such that*

$$\Lambda \cap (\mathbb{R} \times [0, 1) \times \mathbb{R}^2) = \{(m + t_k, 0, j + \mu_{k,m}, k) : j, k, m \in \mathbb{Z}\}$$

Proof. By the result of Lemma 4.5, we have the identities (4.4) and (4.5). Let $T = \{t_k, k \in \mathbb{Z}\} \subset [0, 1)$ where $t_k, k \in \mathbb{Z}$, are the numbers appearing in (4.4). Let $s_1, s_2 \in T$ with $s_1 < s_2$. Consider the half-open squares $C = (s_1, 0) + [0, 1]^2$ and $C' = (s_1, 0) + ((0, 1] \times [0, 1))$. Then we know that $\Gamma(C) + [0, 1]^2$ and $\Gamma(C') + [0, 1]^2$ both tile \mathbb{R}^2 . Let $P_0 = \{(s_1, y) : 0 \leq y < 1\}$ and $P_1 = \{(s_1 + 1, y) : 0 \leq y < 1\}$. Note that $\Gamma(P_0) = \Gamma(\{(s_1, 0)\})$. Moreover,

$$\Gamma(C) = \Gamma(P_0) \cup \Gamma(C \setminus P_0), \quad \Gamma(C') = \Gamma(C' \setminus P_1) \cup \Gamma(P_1)$$

and since $C \setminus P_0 = C' \setminus P_1$, $\Gamma(P_0) = \Gamma(P_1)$. We have

$$T_{C'}(\Gamma(P_1)) \subset \{(s_1 + 1, y), 0 \leq y < 1\}$$

but since $(s_2, 0) \in C'$, we must have $T_{C'}(\Gamma(P_1)) = (s_1 + 1, 0)$ by Lemma 4.4. Since

$$\Gamma(P_0) = \{(j + \mu_{k,0}, k) : j, k \in \mathbb{Z}, t_k = s_1\}$$

and $\pi_2(\Gamma(P_0)) = \pi_2(\Gamma(P_1))$, where π_2 is the projection to the second coordinate, we have

$$\Gamma(\{(1 + s_1, 0)\}) = \Gamma(P_1) = \{(j + \mu_{k,1}, k) : j, k \in \mathbb{Z}, t_k = s_1\}.$$

for some constants $\mu_{k,1}$ with $0 \leq \mu_{k,1} < 1$ using Proposition 3.2. Applying this argument to $s_1 = 0$ and $s_2 = t$, we obtain that

$$\Lambda \cap (\{1\} \times [0, 1) \times \mathbb{R}^2) = \{(j + \mu_{k,1}, k) : j, k \in \mathbb{Z}, t_k = 0\}.$$

Similar arguments applied to $s_1 = s$ and $s_2 = 1$ show that, for any $s \in T$, we have

$$\Lambda \cap (\{s + 1\} \times [0, 1) \times \mathbb{R}^2) = \{(j + \mu_{k,1}, k) : j, k \in \mathbb{Z}, t_k = s\}.$$

and that $\Lambda \cap (\{s + 1\} \times [0, 1) \times \mathbb{R}^2)$ is empty if $s \in [0, 1) \setminus T$. The same idea can also be used to show the existence of constants $\mu_{k,-1}$ with $0 \leq \mu_{k,-1} < 1$ such that

$$\Lambda \cap (\{s - 1\} \times [0, 1) \times \mathbb{R}^2) = \begin{cases} \{(j + \mu_{k,-1}, k) : j, k \in \mathbb{Z}, t_k = s\}, & s \in T, \\ \emptyset, & s \in [0, 1) \setminus T. \end{cases}$$

and, more generally using induction, that, for any $m \in \mathbb{Z}$, we can find constants $\mu_{k,m}$ with $0 \leq \mu_{k,m} < 1$ such that

$$\Lambda \cap (\{s + m\} \times [0, 1) \times \mathbb{R}^2) = \begin{cases} \{(j + \mu_{k,m}, k) : j, k \in \mathbb{Z}, t_k = s\}, & s \in T, \\ \emptyset, & s \in [0, 1) \setminus T. \end{cases}$$

This proves our claim. □

We can now complete the proof of the main result of this section which gives a characterization for the subsets Λ of \mathbb{R}^4 with the property that the associated set of time-frequency shifts applied to the window $\chi_{[0,1]^2}$ yields an orthonormal basis for $L^2(\mathbb{R}^2)$.

Proof of Theorem 1.4. It follows from Lemma 4.4 that either all $T_{[0,1]^2}(\lambda_1, \lambda_2)$, $(\lambda_1, \lambda_2) \in \Gamma([0, 1]^2)$ are either of the form $\{(t, 0)\}$ or all are of the form $\{(0, t)\}$ with some $t \neq 0$. In the first case, we deduce from Lemma 4.6 that

$$\Lambda \cap (\mathbb{R} \times [0, 1) \times \mathbb{R}^2) = \{(m + t_k, 0, j + \mu_{k,m}, k) : j, k, m \in \mathbb{Z}\}$$

for certain numbers t_k and $\mu_{k,m}$ in the interval $[0, 1)$. We now show that Λ will be of the first of the two possible forms given in the theorem. (Similarly, the second form follows from the second case of Lemma 4.6).

Letting $C = [0, 1]^2$ and $C' = [0, 1) \times (0, 1]$, we note that both $\Gamma(C) + [0, 1]^2$ and $\Gamma(C') + [0, 1]^2$ tile \mathbb{R}^2 but $\Gamma((0, 1)^2)$ is empty. Hence, $\Gamma(C') = \Gamma(\{(x, 1) : 0 \leq x < 1\})$. It means that any set $T_{C'}(\lambda_1, \lambda_2)$ with $(\lambda_1, \lambda_2) \in \Gamma(C')$ is of the form $\{(t, 1)\}$ for some $t = t(\lambda_1, \lambda_2)$ with $0 \leq t < 1$. We now have two possible cases: either the cardinality of $T_{C'}(\Gamma(C'))$ is larger than one or equal to one. In the first case, we

can find two distinct elements of $T_{C'}(\Gamma(C'))$ and we can then replicate the proof of Lemma 4.6 to obtain that

$$\Lambda \cap (\mathbb{R} \times [1, 2) \times \mathbb{R}^2) = \{(m + t_k, 1, j + \mu_{k,m,1}, k) : j, k \in \mathbb{Z}\}.$$

In the other case, $T_{C'}(\Gamma(C')) = \{(t_1, 1)\}$ for some t_1 with $0 \leq t_1 < 1$. If we translate C' horizontally and use the same argument as in the proof of Lemma 4.6, we see that

$$\Lambda \cap (\mathbb{R} \times [1, 2) \times \mathbb{R}^2) = \{(m + t_1, 1)\} \times \Lambda_{m,1},$$

where $\Lambda_{m,1}$ is a spectrum for the unit square $[0, 1]^2$. This last property is equivalent to $\Lambda_{m,1} + [0, 1]^2$ being a tiling of \mathbb{R}^2 by the result in [LRW].

We can then prove the theorem inductively by translating the square C' in the vertical direction using integer steps. \square

REFERENCES

- [AFK] G. ASCENSI, H. G. FEICHTINGER AND N. KAIBLINGER, *Dilation of the Weyl Symbol and the Balian-Low theorem*, Trans. Amer. Math. Soc., 366, (2014), 3865–3880.
- [DS] X.-R. DAI AND Q.-Y. SUN, *The abc-problem for Gabor systems*, preprint, <http://arxiv.org/abs/1304.7750>
- [Fu] B. FUGLEDE, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal., 16 (1974), 101–121.
- [G] K. GRÖCHENIG, *Foundations of time-frequency analysis*, Appl. Num. Harmon. Anal., Birkhäuser, Boston, Basel, Berlin, 2001.
- [GS] K. GRÖCHENIG AND J. STÖCKLER, *Gabor frames and totally positive functions*, Duke Math. J., 162 (2013), 1003–1031.
- [Ga] J.-P. GABARDO, *Tight Gabor frames associated with non-separable lattices and the hyperbolic Secant*, Acta Appl. Math., 107 (2009), 49–73.
- [Gab] D. GABOR, *Theory of communication*, J. Inst. Elec. Eng. (London), 93 (1946), 429V457.
- [GH] Q. GU AND D. HAN, *When a characteristic function generates a Gabor frame*, Appl. Comp. Harm. Anal., 24 (2008), 290–309.
- [J1] A. JANSSEN, *Zak Transform with few zeros and the ties*, in: Advances in Gabor Analysis, in: Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2003, 31–70.
- [J2] A. JANSSEN, *On generating tight Gabor frames at critical density*, J. Fourier Anal. Appl. 9 (2003), 175–214.
- [K] M. KOLOUNTZAKIS, *The study of translational tiling with Fourier analysis*, Fourier Analysis and Convexity, Appl. Numer. Harmon. Anal., Birkhäuser Boston (2004), 131–187.
- [L] Y. LYUBARSKII, *Frames in the Bargmann space of entire functions. Entire and subharmonic functions*, Adv. Soviet Math., 11, 167–180, Amer. Math. Soc., Providence, RI, 1992.
- [Li] J.-L. LI, *On characterization of spectra and tilings*, J. of Funct. Anal., 213 (2004) 31–44.
- [LiW] Y.M. LIU AND Y. WANG, *The uniformity of non-uniform Gabor bases*, Adv. Comput. Math., 18 (2003), 345–355.
- [LRW] J. LAGARIAS, J. REEDS AND Y. WANG, *Orthonormal bases of exponentials for the n-cubes*, Duke Math. J., 103 (2000), 25–37.
- [RS] J. RAMANATHAN AND T. STEGER, *Incompleteness of sparse coherent states*, Appl. Comp. Harm. Anal. 2 (1995), 148–153.

- [SW] K. SEIP AND R. WALLSTEN, *Density theorems for sampling and interpolation in the Bargmann-Fock space, II*, J. Reine Angew. Math., 429 (1992), 107–113.
- [T] T. TAO, *Fuglede’s conjecture is false in 5 or higher dimensions*, Math. Res. Letter, 11 (2004), 251–258.

E-mail address: gabardo@mcmaster.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, L8S 4K1, CANADA

E-mail address: cklai@sfsu.edu

DEPARTMENT OF MATHEMATICS, SAN FRANCISCO STATE UNIVERSITY, 1600 HOLLOWAY AVE., SAN FRANCISCO, CA 94132.

E-mail address: yangwang@ust.hk

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG